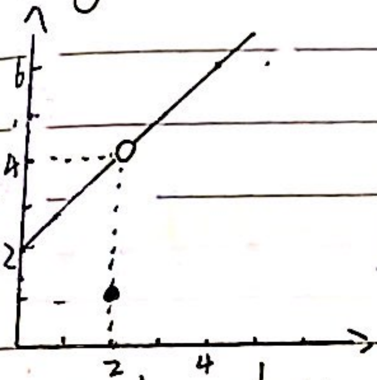
instant: is an infinitely small amount of time Δt (or change in t is) = 0

average rate of change: average rate of change over an interval is the slope 斜率 between the endpoints of the interval.

The instantaneous rate of change of a function is the slope of the line tangent 切线 to the function at that point.

We can approximate instantaneous rates of change by finding average rates of change over small intervals.

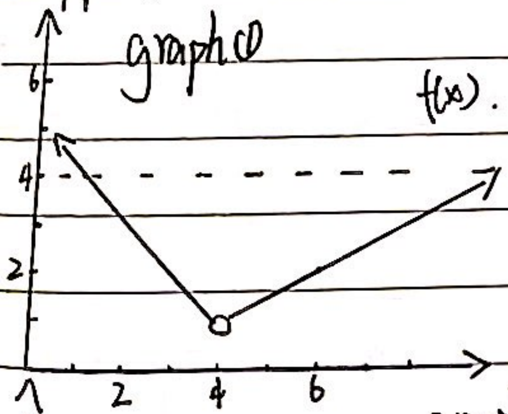


As the x values get closer and closer to 2, the y values are getting closer and closer to 4 → As x approaches 2, f(x) approaches 4.

~~At~~ Notation (limit statement): $\lim_{x\to 2} f(x)=4$

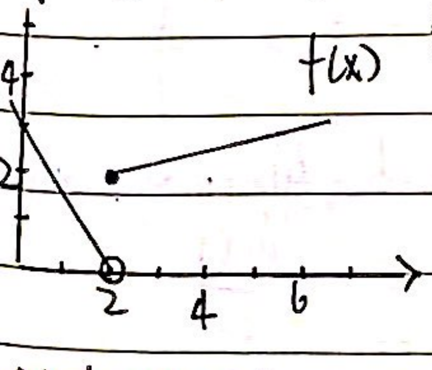
Verbal Statement: the limit of f(x) as x approaches 2 is 4.

approaches: closer and closer   limit 和 f(x) 没关系.



left side: $\lim_{x\to 4^-} f(x)$

right side $\lim_{x\to 4^+} f(x)$

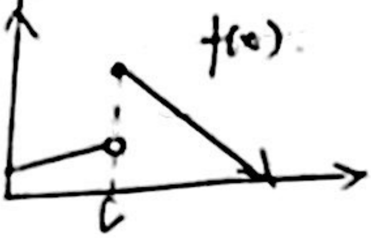


$\lim_{x\to 2^-} f(x)=2$ $\lim_{x\to 2^+} f(x)=2$

If $\lim_{x\to c^-} f(x)=\lim_{x\to c^+} f(x)$ then $\lim_{x\to c} f(x)$ exists!

graph① have $\lim_{x\to c} f(x)$ and graph② do not limits that fail to exist

1. Jump

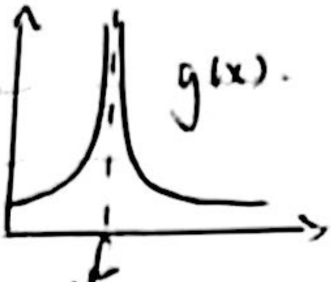


example:

$\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$

$f(x)\begin{cases} x, & x<2 \\ 6-x, & x>2 \end{cases}$

2. Unbounded



$\lim_{x\to c} g(x) = +\infty$ $\quad g(x)=2+\frac{1}{(x-2)^2}$

and $\lim_{x\to\infty} f(x)$ is also unbounded.

3. Oscillating 振荡

beware of technology $\quad h(x)=\sin\left(\frac{1}{x}\right)$

点过于分散，计算机

无法做图

If $\lim_{x\to c} f(x)=L$ $\quad \lim_{x\to c} g(x)=M$ $\quad$ both of them need to exist

$b\cdot \lim_{x\to c} f(x) = \lim_{x\to c} [b f(x)] = b\cdot L$

$\lim_{x\to c} f(x) \pm \lim_{x\to} g(x) = \lim_{x\to c} [f(x)\pm g(x)] = L\pm M$

$\lim_{x\to c} f(x)\cdot \lim_{x\to c} g(x) = \lim_{x\to c} [f(x)\, g(x)] = L\cdot M$

$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{L}{M}$ $\quad (M\neq 0)$

$\lim \frac{ac}{ab} = \lim \frac{c}{b}$ ← the graph may be different, but limits are same.

A method: like $\frac{x-6}{\sqrt{x-5}-1} = \frac{x-6}{\sqrt{x-5}-1}\cdot\frac{\sqrt{x-5}+1}{\sqrt{x-5}+1} = \frac{(x-6)(\sqrt{x-5}+1)}{x-5-1}$

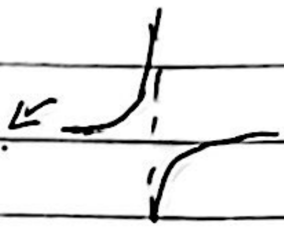
If $g(x) \le f(x) \le h(x)$ for all values of x in an open interval containing c, except possibly at c itself,

$$\lim_{x\to c} g(x) = L = \lim_{x\to c} h(x).$$

$$\to \lim_{x\to c} f(x) = L.$$

Types of discontinuities include...

- removable discontinuities. (hole).
- jump discontinuities.
- discontinuities due to vertical asymptotes. ←



When function $f(x)$. is continuous at $x=c$

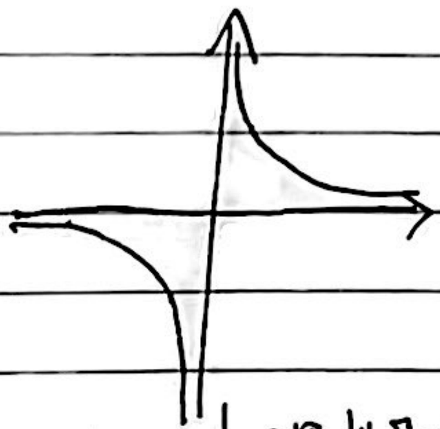
① $f(c)$ must exist.

② $\lim_{x\to c} f(x)$ must exist.

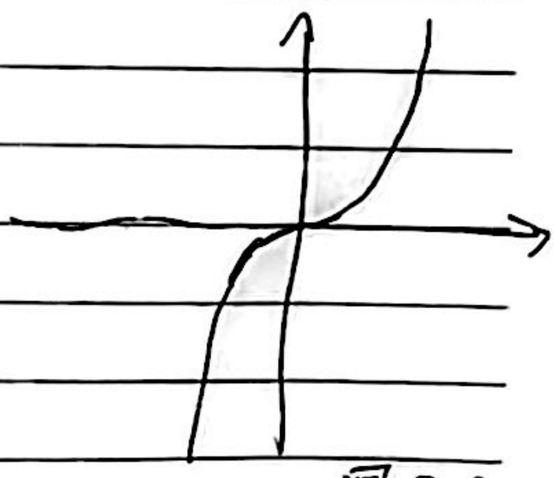
③ $f(c)$ must equal $\lim_{x\to c} f(x)$.

A function is continuous on an interval if the function is ~~contine~~ continuous at each point in the interval.

polynomial 多项式



rational 理性函数.



power 幂函数

exponential 指數函數 logarithmic. trigonometric

$\lim\limits_{x\to\infty} \frac{c}{x^n} = 0$. example: $\lim\limits_{x\to\infty} \frac{3x^3 - 1000x^2 + 5x - 7}{3x^3}$

$1 - \frac{1000x^2}{3x^3} + \frac{5x}{3x^3} + \frac{-7}{3x^3}$

$1 - \frac{1000}{3x} + \frac{5}{3x^2} - \frac{7}{3x^3}$.

$\to \lim\limits_{x\to\infty} 1 - \frac{1000}{3}\lim\limits_{x\to\infty}\frac{1}{x} + \frac{5}{3}\lim\limits_{x\to\infty}\frac{1}{x^2} - \frac{7}{3}\lim\limits_{x\to\infty}\frac{1}{3x^3}$.

$1 - 0 + 0 - 0 = 1$.

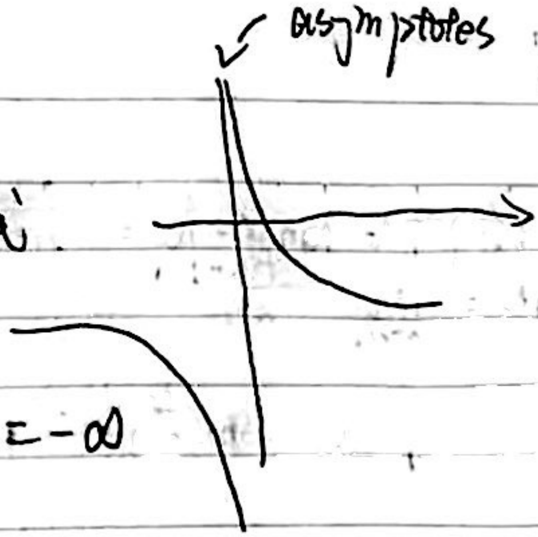
$\lim\limits_{x\to\infty} \frac{\text{多項式}}{\text{多項式}}$ example $\lim\limits_{x\to\infty} \frac{x - 8x^4}{7x^4 + 5x^3 + 2000x^2 - 6}$ 指數最大.

$$\frac{\frac{x - 8x^4}{-8x^4} \times (-8x^4)}{\frac{7x^4 + 5x^3 + 2000x^2 - 6}{7x^4} \times (7x^4)}.$$

No

Date

vertical asymptotes. 垂直渐近线



$\lim\limits_{x\to c^-} f(x) = \infty$. $\lim\limits_{x\to c^+} f(x) = -\infty$

$x = c$. vertical asymptote.

分数只看最高次向

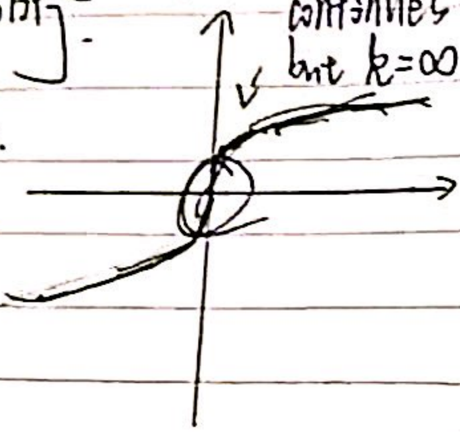
Rate of change (斜率): $\tan\alpha = k = \frac{f(b)-f(a)}{b-a}$.

instantaneous rate = $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ or $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

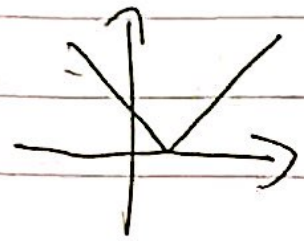
derivative of $f(x) = f'(x) = \lim \frac{f(x+h)-f(x)}{h}$ or $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ (导数).

Differentiability (可微性) implies continuity, but continuity ~~de~~ does not guarantee differentiability.

example ①.



②.



③.

$f(x)=x^n$. $\rightarrow$ $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^n - x^n}{h}$ —— $= nx^{n-1}$

$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)x^{n-2}}{2}h^2 + \cdots + h^n$.

The derivative of a constant function (常数函数, 如 $y=3$) is 0.

The sum or difference of two differentiable functions is differentiable and is the sum or difference of their derivatives.

If $f(x)$ is differentiable, c (常数) $f(x)$ is also differentiable.

example. $A(x) = \frac{2}{x} + f(x)$ $\quad 2x^{-1} \to nx^{n-1} = -2x^{-2}$

$A'(x) = -2x^{-2} + f'(x)$

$\frac{d}{dx}[\sin x] = \cos x.$ $\qquad \frac{d}{dx}[\cos x] = -\sin x.$

$\frac{d}{dx}[e^x] = e^x$ $\qquad \frac{d}{dx}[\ln x] = \frac{1}{x}$

$$\frac{d}{dx}[\cos x] = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \frac{\cos x \cdot \cos h - \sin x \cdot \sin h - \cos x}{h}$$

$$= \frac{\cos x(\cos h - 1) - \sin x \cdot \sin h}{h}.$$

$$= \cos x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h \to 0} \frac{\sin h}{h}$$

$$= \cos x \cdot 0 - \sin x \cdot 1$$

$$= -\sin x$$

The product of two differentiable functions f and g is itself differenti
Moreover, the derivative of fg is the first function times the derivative
the second, plus the second function times the derivative of the first fun

$$\frac{d}{dx}[f(x)g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$\frac{d}{dx}[\tan x] = \sec^2 x$ $\frac{d}{dx}[\cot x] = -\csc^2 x$.

$\frac{d}{dx}[\sec x] = \sec x \cdot \tan x$ $\frac{d}{dx}[\csc x] = -(\csc x \cdot \cot x)$.

$\sec = \frac{1}{\cos x}$ $\csc \cancel{x} = \frac{1}{\sin}$ $\cot = \frac{1}{\tan}$

$a = e^{\ln a}$.

example: $f(x) = (x^2+1)^{(2-3x)}$.

$= e^{\ln (x^2+1)^{(2-3x)}}$.

$= e^{(2-3x)\ln(x^2+1)}$.

$f'(x) = e^{(2-3x)\ln(x^2+1)} \cdot -3\cancel{x}\ln(x^2+1) \cdot 2x(2-3x)$.

The Chain Rule

If $y=f(u)$ is a differentiable function of u and $u=g(x)$ is a differentiable function of x, then $y=f(g(x))$ is a differentiable function and

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$ or equivalently $$y'=f'(g(x))\cdot g'(x)$$

If $y=[u(x)]^n$ where "u" is a differentiable function of x and n is a rational number, then $$\frac{dy}{dx}=n[u(x)]^{n-1}\cdot\frac{du}{dx}.$$ X 不用记，由 the chain rule 推得

for example: $x^4-x^2\cdot y+y^4=24$. find $\frac{dy}{dx}$.

$$=4x^3-(2x\cdot y+x^2\cdot\frac{dy}{dx})+4y^3\cdot\frac{dy}{dx}=0.$$

$$=4x^3-2xy-x^2\frac{dy}{dx}+4y^3\cdot\frac{dy}{dx}$$

$$\frac{dy}{dx}(-x^2+4y^3)=2xy-4x^3$$

$$\frac{dy}{dx}=\frac{2xy-4x^3}{-x^2+4y^3}$$

$(x^2+y^2)^2=\frac{25}{4}xy^2$ find (1,2) tangent.

$$\frac{d(x^2+y^2)^2}{dx}=\frac{d(\frac{25}{4}xy^2)}{dx}.$$ Chain Rule.

$$2(x^2+y^2)\cdot(2x+2y)=\frac{25}{4}y^2+\frac{25}{4}x\cdot2y\cdot\frac{dy}{dx}$$

$$2(1^2+2^2)\cdot(2\times1+2\times2\cdot\frac{dy}{dx})=\frac{25}{4}2^2+\frac{25}{4}\times1\times2\times2\frac{dy}{dx}$$

The Derivatives of Inverse Functions.

Let f be a function that is differentiable on an interval I. If f has an inverse function f^{-1} is differentiable at any x for ~~each~~ which $f'(f^{-1}(x)) \neq 0$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

$y = \sin^{-1}(u) \rightarrow \frac{dy}{dx} = \frac{u'}{\sqrt{1-u^2}}$ $\quad$ $y = \cos^{-1}(u)$ $\frac{dy}{dx} = -\frac{u'}{\sqrt{1-u^2}}$

$y = \tan^{-1}(u)$ $\frac{dy}{dx} = \frac{u'}{1+u^2}$ $\quad$ $y = \cot^{-1}(u)$ $\frac{dy}{dx} = -\frac{u'}{1+u^2}$

$y = \sec^{-1}(u)$ $\frac{dy}{dx} = \frac{u'}{|u|\sqrt{u^2-1}}$ $\quad$ $y = \csc^{-1}(u)$ $\frac{dy}{dx} = -\frac{u'}{|u|\sqrt{u^2-1}}$

the units of derivatives are $\frac{y \text{ units}}{x \text{ units}}$.

movement: $s''(t) = v'(t) = a(t)$

example: Find the equation of the line tangent to $f(x)=e^x$ at $x=0$. Use your result to approximate $\sqrt[3]{e}$.

point-slope: $y-y_0 = k(x-x_0)$.

$e^x = e^0 = 1$. $f'(x) = e^x = 1$.

$y-1 = 1(x-0)$

$y-1 = x$

$y = x+1$. $\sqrt[3]{e} = e^{\frac{1}{3}}$. $x=\frac{1}{3}$.

代入: $y = x+1$

$= \frac{1}{3}+1$

$= \frac{4}{3}$

$\sqrt[3]{e} \approx \frac{4}{3}$.

L'Hospital's Rule.

Let f and g be functions that are differentiable on an open interval (a,b) containing c, except possibly at c itself. Assume $g'(x) \neq 0$ for all x in (a,b) except possibly at c itself.

If $\lim_{x\to c} f(x)=0$ and $\lim_{x\to c} g(x)=0$ then:

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

The Mean Value Theorem.

If a function f is continuous on the closed interval $[a,b]$ and differentiable the open interval (a,b), then there exists at least one (may be more than one) number c in the open interval (a,b) where the instaneous rate of change equals the average rate of change over the interval.

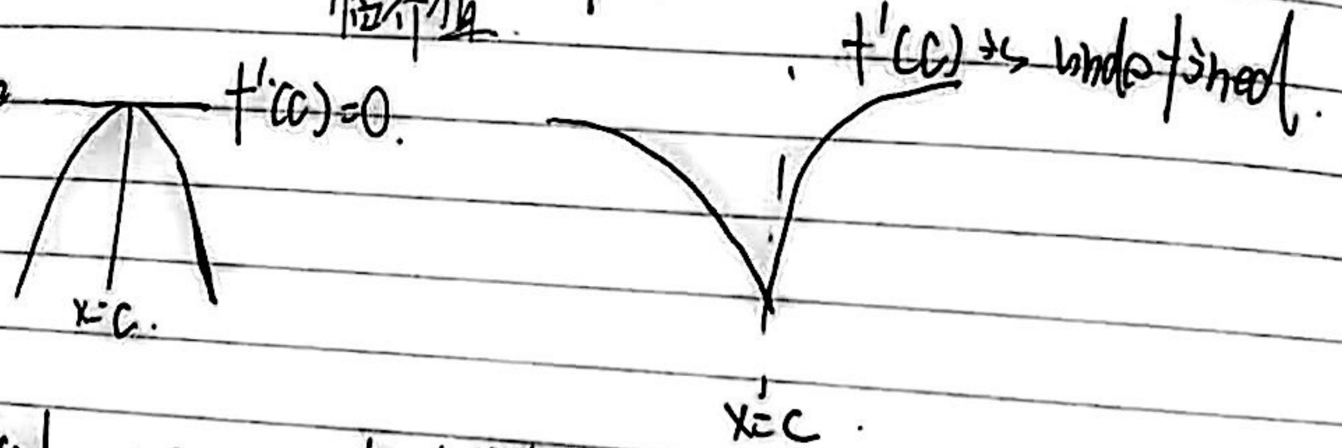
$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

For the constant function, the highest value equals to the lowest value.

when $f(c) \geq f(x)$. x is any real number $f(c)$ is the maximum of f.

Let f be defined at c. If $f'(c)=0$ or if f is not differentiable at c, then c is a critical number of f.
临界值.

like



critical points as local/global maximum / minimum, or none of the above

After use the first derivative to find possible critical point, use second derivative positive or negative to ~~find~~ classify.

inflection point 拐点

example: $5x^2-3y^2=20$.

derivative: $10x-6y\frac{dy}{dx}=0$.

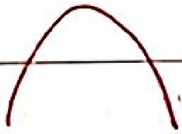
$$\frac{dy}{dx}=\frac{5x}{3y}$$

critical points ① $\frac{dy}{dx}=0$. $x=0$. $-3y^2\neq 20$.

② $\frac{dy}{dx}$ undefined $3y=0$ $y=0$ $x=\pm 2$.

$(2,0)$ $(-2,0)$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right).$$ 即 $f''(x)$.

Concave done 是 .

Concave up. 是